ISRAEL JOURNAL OF MATHEMATICS 93 (1996), 113–124

CONSISTENCY STRENGTH OF THE AXIOM OF FULL REFLECTION AT LARGE CARDINALS

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ABSTRACT

We prove that the Axiom of Full Reflection at a measurable cardinal is equiconsistent with the existence of a measurable cardinal. We generalize the result also to larger cardinals such as strong or supercompact cardinals.

1. Introduction

It has been proved in [JS90] that the Axiom of Full Reflection at an n-Mahlo cardinal is equiconsistent with a Π^1_n-indescribable cardinal and in [JW94] that consistency of the Axiom of Full Reflection at a measurable cardinal follows from consistency of a coherent sequence of measures with a repeat point. It was conjectured in [JW94] that the two principles are actually equiconsistent. However we prove that Full Reflection at a measurable cardinal can be obtained surprisingly from only one measure. Furthermore the method also generalizes to larger cardinals such as strong or supercompact cardinals. Hence we can conclude that the Axiom of Full Reflection at large cardinals weaker than measurable,

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Received May 11, 1994

e.g. as n-Mahlo, does push the consistency strength up, but does not push the consistency strength up at measurable or larger cardinals.

To state the main theorem let us review the basic definitions and facts. If S is a stationary subset of a regular uncountable cardinal κ then **the trace of** S is the set

$$\mathrm{Tr}\,(S) = \{\alpha < \kappa\colon S \cap \alpha \text{ is stationary in } \alpha\}$$

and we say that S **reflects at** $\alpha \in \mathrm{Tr}\,(S)$. If S and T are both stationary, we define

$$S < T \text{ if } \alpha \in \mathrm{Tr}\,(S) \text{ for almost all } \alpha \in T,$$

and say that S **reflects fully** in T. (Throughout the paper, "for almost all" means "except for a nonstationary set of points"). It can be proved that this relation is a well-founded partial ordering (see [JW94] or [J84]). Let

$$\mathrm{Reg}(\kappa) = \{\alpha < \kappa\colon \alpha \text{ is a regular cardinal}\},$$
$$\mathrm{Sing}(\kappa) = \kappa \setminus \mathrm{Reg}(\kappa).$$

The order $o(S)$ of a stationary set of regular cardinals is defined as the rank of S in the relation $<$:

$$o(S) = \sup\{o(T) + 1\colon T \subseteq \mathrm{Reg}(\kappa) \text{ is stationary and } T < S\}.$$

For a stationary set T such that $T \cap \mathrm{Sing}(\kappa)$ is stationary define $o(T) = -1$. **The order of** κ is then defined as

$$o(\kappa) = \sup\{o(S) + 1\colon S \subseteq \kappa \text{ is stationary}\}.$$

Note that the order $o(\kappa)$ provides a natural generalization of the Mahlo hierarchy: κ is exactly $o(\kappa)$-Mahlo if $o(\kappa) < \kappa^+$ and greatly Mahlo if $o(\kappa) \geq \kappa^+$.

We say that a stationary set S **reflects fully** at regular cardinals if for any stationary set T of regular cardinals $o(S) < o(T)$ implies $S < T$.

AXIOM OF FULL REFLECTION AT κ: *Every stationary subset of κ reflects fully at regular cardinals.*

Notice that the axiom presents in a sense the maximal possible amount of reflection of stationary subsets of κ at regular cardinals.

Now we are able to state the main theorem:

THEOREM: *Let $\phi(\kappa)$ be one of the following principles:*

(i) *κ is measurable,*
(ii) *the Mitchell order of κ is κ^{++},*
(iii) *κ is n-strong,*
(iv) *κ is strong,*
(v) *κ is κ^{+n}-supercompact,*
(vi) *κ is supercompact.*

Assume that V satisfies GCH and $\phi(\kappa)$, then there is a model where GCH, the Axiom of Full Reflection at κ, and $\phi(\kappa)$ hold.

The case (ii) has been actually proved in [JW94]: it has been proved in the paper that if $\vec{\mathcal{U}}$ is a coherent sequence of measures then there is a forcing notion $P_{\kappa+1}$ that preserves any repeat point of $\vec{\mathcal{U}}$ on κ. If $o^{\mathcal{U}}(\kappa) = \kappa^{++}$ then there are κ^{++} repeat points on κ and it is not difficult to see that the Mitchell order of κ is κ^{++} in the generic extension by $P_{\kappa+1}$. Thus we will work only on cases (i) and (iii)–(vi).

2. Proof of the theorem

The proof should be self-contained, although a knowledge of [JW94] would be helpful.

Assume that V satisfies GCH and $j\colon V \to M$ is an elementary embedding such that $\mathrm{crit}(j) = \kappa$ and $V \cap {}^{\kappa}M \subseteq M$. We will define a forcing $P_{\kappa+1}$ that will work in all cases (i), (iii)–(vi). $P_{\kappa+1}$ will be an Easton support iteration of $\langle Q_\lambda ; \lambda \leq \kappa \rangle$; Q_λ will be nontrivial only for λ Mahlo, and in that case it will be an iteration (defined in $V(P_\lambda)$) of length λ^+ with $< \lambda$ support of forcing notions shooting clubs through certain stationary sets $X \subseteq \lambda$ always with the property that $X \supseteq \mathrm{Sing}(\lambda)$. This will guarantee Q_λ to be essentially $< \lambda$-closed (i.e. it will have a $< \lambda$-closed dense subset). Consequently Q_λ will be λ^+-c.c., P_λ will be λ-c.c., and the factor iteration $P_{\lambda+1,\kappa+1}$ above λ will be essentially λ-closed. Therefore $P_{\kappa+1}$ will preserve cardinals, cofinalities, and GCH.

Consider an iteration Q of $\langle \mathrm{CU}(\dot{X}_\alpha)\colon \alpha < l(Q) \rangle$ with $< \lambda$ support, where $\mathrm{CU}(\dot{X}_\alpha)$ denotes the forcing shooting a club in $V(P_\lambda * Q \restriction \alpha)$ through a stationary subset $\dot{X}_\alpha$ of λ containing $\mathrm{Sing}(\lambda)$. In that case we say that Q is **an iteration of order 0**. Since $Q \restriction \alpha$ is essentially $< \lambda$-closed, conditions in $\mathrm{CU}(\dot{X}_\alpha)$ can be taken in $V(P_\lambda)$ rather than in $V(P_\lambda * Q \restriction \alpha)$. So Q can be considered to be a set of sequences of closed bounded subsets of λ in $V(P_\lambda)$. Since P_λ is λ-c.c. there is an appropriate P_λ-name for Q of cardinality λ if $l(Q) < \lambda^+$, and of cardinality λ^+ if $l(Q) = \lambda^+$. Let $\tilde{Q}$ be another iteration of $\langle \mathrm{CU}(\dot{Y}_\gamma)\colon \gamma < l(\tilde{Q})\rangle$ of order 0. We say that Q is an subiteration of $\tilde{Q}$ if there is an injection function $\pi\colon l(Q) \to l(\tilde{Q})$ sending α to γ_α inducing an embedding of Q into $\tilde{Q}$ such that $\dot{X}_\alpha$ is an equivalent name to $\dot{Y}_{\gamma_\alpha}$ with respect to the induced embedding of $Q \restriction \alpha$ into $\tilde{Q}$. Notice that the sequence $\langle \gamma_\alpha\colon \alpha < l(Q)\rangle$ does not have to be increasing. Any Q-name can be considered to be a $\tilde{Q}$-name via the induced embedding; $\tilde{Q}$ is actually isomorphic to an iteration of order 0 in the form $Q * R$.

We will need to estimate (in $V(P_\lambda)$) the number of iterations of order 0 and length $< \lambda^+$. Each such iteration is a set of sequences with $< \lambda$ support of bounded subsets of λ. Therefore it is easy to see that the number is at most $2^\lambda = \lambda^+$.

For any iteration Q of order $\delta + 1$ we will define certain filters $F^Q_{\lambda,\delta}$ on λ in $V(P_\lambda * Q)$. Simultaneously by induction on β and $l(Q)$ we define Q to be **an iteration of order** β if it is the iteration of $\langle CU(\dot{X}_\alpha)\colon \alpha < l(Q)\rangle$ with $< \lambda$-support such that $l(Q) < \lambda^+$ and for all $\alpha < l(Q)$:

$$P_\lambda * Q \restriction \alpha \Vdash \text{“}\mathrm{Sing}(\lambda) \subseteq \dot{X}_\alpha \text{ and } \dot{X}_\alpha \in F^{Q\restriction\alpha}_{\lambda,\delta} \text{ for all } \delta < \beta.\text{”}$$

Let us call such an assignment $Q \mapsto F^Q_{\lambda,\delta}$ a filter system $F_{\lambda,\delta}$. $F_{\lambda,\delta}$ will be defined for all $\delta < \Theta(\lambda)$ where $\Theta(\lambda)$ will be specified later. The filter systems will have among others the property that $F^{Q\restriction\alpha}_{\lambda,\delta} \subseteq F^Q_{\lambda,\delta}$.

Q_λ is then defined in $V(P_\lambda)$ to be an iteration of length λ^+ such that for all $\alpha < \lambda^+$ $Q_\lambda \restriction \alpha$ is an iteration of order $\Theta(\lambda)$, and all potential names for stationary subsets of λ are used cofinally many times.

If F is a filter on a set A we say that $X \subseteq A$ is **F-positive** if $A \setminus X \notin F$, and **F-thin** if $A \setminus X \in F$.

It remains to find the filter systems $F_{\lambda,\delta}$ (working in $V(P_\lambda)$). We require that for any iteration Q of order $\delta + 1$ the following is satisfied:

(i) If Q' is an subiteration of Q then

$$F^{Q'}_{\lambda,\delta} = F^Q_{\lambda,\delta} \cap V(P_\lambda * Q'),$$

(ii) $P_\lambda * Q \Vdash$ “$F^Q_{\lambda,\delta} \supseteq \mathrm{Club}(\lambda)$ is a proper filter,

$$\forall S \subseteq \mathrm{Sing}(\lambda) \text{ stationary: } \mathrm{Tr}\,(S) \in F^Q_{\lambda,\delta},$$
$$\forall S \subseteq \lambda\colon (\exists \gamma < \delta\colon S \text{ is } F^Q_{\lambda,\gamma}\text{-positive}) \Rightarrow \mathrm{Tr}\,(S) \in F^Q_{\lambda,\delta},\text{”}$$

(iii) $P_\lambda * Q \Vdash$ "$\forall S \subseteq \mathrm{Reg}(\lambda) : (\forall \gamma < \delta$: S is $F^Q_{\lambda,\gamma}$-thin$) \Rightarrow \kappa \setminus \mathrm{Tr}\,(S) \in F^Q_{\lambda,\delta}$."
Moreover we require that
(iv) there is an iteration Q of order $\delta + 1$, a $P_\lambda * Q$-name $\dot{X}$ for a subset of λ and $p * q \in P_\lambda * Q$ so that

$$p * q \Vdash_{P_\lambda * Q} \text{"}\dot{X} \text{ is } F^Q_{\lambda,\gamma}\text{-thin for all } \gamma < \delta,\text{" but}$$
$$p * q \Vdash_{P_\lambda * Q} \text{"}\dot{X} \text{ is } F^Q_{\lambda,\delta}\text{-positive."}$$

By induction on δ choose a filter system $F_{\lambda,\delta}$ as long as there is such a filter system with properties (i)–(iv). Since the number of iterations Q of length $< \lambda^+$ with $< \lambda$-support shooting closed unbounded subsets of λ is $\leq \lambda^+$ and since $F_{\lambda,\delta}$ is by (iv) different from all $F_{\lambda,\gamma}$ $(\gamma < \delta)$, this process must eventually stop after a number of steps $\Theta(\lambda) < \lambda^{++}$.

Apply this process by induction on all $\lambda < \kappa$ defining an iteration P_κ below κ. Put $P_{\kappa+1} = (jP_\kappa) \restriction (\kappa+1)$. Note that $P_{\kappa+1} = P_\kappa * Q_\kappa$ where Q_κ is an iteration of length κ^+ with $< \kappa$-support, given by certain filter systems $F_{\kappa,\delta}$ $(\delta < \Theta = \Theta(\kappa))$.

We claim that

$$V(P_{\kappa+1}) \models \text{"Full Reflection at } \kappa\text{"}$$

and that the embedding j can in many cases be lifted onto $V(P_{\kappa+1})$.

Let us define F_j in V similarly as in [JW94] to be a Θ-th filter system on κ:

By induction on $l(Q)$ say that Q, the iteration of $\langle CU(\dot{X}_\alpha)\colon \alpha < l(Q)\rangle$, is an iteration of order $\Theta + 1$ w.r.t. F_j if it is an iteration of order Θ and for all $\alpha < l(Q)$

$$P_\kappa * Q \restriction \alpha \Vdash \text{"}\dot{X}_\alpha \in F_j^{Q\restriction\alpha}.\text{"}$$

If Q is an iteration of order $\Theta + 1$ w.r.t. F_j, $\dot{X}$ a $P_\kappa * Q$-name, $p * q \in P_\kappa * Q$, we define $p * q \Vdash$ "$\dot{X} \in F_j^Q$" if

$$(1) \qquad p \Vdash_{jP_\kappa} \text{"}\forall H \in \mathrm{Gen}_j(Q, G^*)\colon q \in H \Rightarrow [H]^j \Vdash_{jQ} \kappa \in j\dot{X}.\text{"}$$

Here $\mathrm{Gen}_j(Q, G^*)$ and $[H]^j \in jQ$ are defined as follows:

Let G^* be a jP_κ-generic filter over V, $G = G^* \restriction P_\kappa$. Then Q is obviously an subiteration of Q_κ which gives a filter H from $G^* \restriction Q_\kappa$ that is Q-generic over $V[G]$. $\mathrm{Gen}_j(Q, G^*)$ denotes the set of all filters H obtained in this way. We can easily find many $H \in \mathrm{Gen}_j(Q, G^*)$ such that $q \in H$: since Q_κ is an iteration of order Θ such that all potential names are used cofinally many times we can find a sequence of ordinals $\langle \gamma_\alpha\colon \alpha < l(Q)\rangle$ inducing a subiteration embedding of

Q into Q_κ such that all γ_α's are above any given $\beta < \kappa^+$; hence by a density argument there is $r \in G^*$ and such a sequence $\langle \gamma_\alpha : \alpha < l(Q) \rangle$ with the property that $r \restriction \langle \gamma_\alpha : \alpha < l(Q) \rangle = q$.

Represent an $H \in \mathrm{Gen}_j(Q, G^*)$ as $\langle C_\beta : \beta < l(Q) \rangle$ where C_β's are the generic closed unbounded subsets of κ. $[H]^j$ is a sequence of length $j(l(Q))$ defined as follows

$$[H]^j(\gamma) = \begin{cases} C_\beta \cup \{\kappa\} \text{ if } j(\beta) = \gamma, \\ \emptyset \text{ otherwise.} \end{cases}$$

To prove that $[H]^j \in jQ$ all we need is to check inductively that

$$[H]^j \restriction j(\beta) \Vdash_{j(Q \restriction \beta)} \text{“}\kappa \in j\dot{X}_\beta.\text{”}$$

But this immediately follows from the assumption $P_\kappa * Q \restriction \beta \Vdash$ “$\dot{X}_\beta \in F_j^{Q \restriction \beta}$.”

LEMMA 1: *The filter system $F_{\kappa,\Theta} = F_j$ satisfies* (i)–(iii) *with $\delta = \Theta$.*

Proof: (i) Let Q, Q' be two iterations of order $\Theta + 1$; assume π embeds Q into Q' via $\langle \alpha_\delta : \delta < l(Q) \rangle$ as an subiteration. Let $\dot{X}$ be a $P_\lambda * Q$-name for a subset of λ.

Suppose $p * q \in P_\lambda * Q$, $p * q \Vdash_{P_\lambda * Q}$ “$\dot{X} \in F_j^Q$.” We want to prove that

$$p * \pi(q) \Vdash_{P_\lambda * Q'} \text{“}\pi(\dot{X}) \in F_j^{Q'}.\text{”}$$

Let G^* be jP_λ-generic over V with $p \in G^*$, $H' \in \mathrm{Gen}_j(Q', G^*)$ with $\pi(q) \in H'$. Then the embedding of Q' into $(jP_\lambda)^\lambda$ induces via π an embedding of Q into $(jP_\lambda)^\lambda$ giving $H \in \mathrm{Gen}_j(Q, G^*)$ such that $q \in H$. Moreover $j\pi$ embeds jQ into jQ' by elementarity, and $(j\pi)([H]^j) \geq [H']^j$. Since $[H]^j \Vdash_{jQ}$ “$\check{\lambda} \in j\dot{X}$” it follows that $[H']^j \Vdash_{jQ'}$ “$\check{\lambda} \in j(\pi \dot{X})$.”

Now suppose $p * q' \in P_\lambda * Q'$ with $p * q' \Vdash_{P_\lambda * Q'}$ “$\pi(\dot{X}) \in F_j^{Q'}$.” Let $q \in Q$ be such that $\pi(q)$ agrees with q' on the set $\{\alpha_\delta : \delta < l(Q)\}$. We claim that $p * q \Vdash_{P_\lambda * Q}$ “$\dot{X} \in F_j^Q$.” Let G^* be jP_λ-generic over V with $p \in G^*$, $H \in \mathrm{Gen}_j(Q, G^*)$ with $q \in H$. We need to prove $[H]^j \Vdash_{jQ}$ “$\check{\lambda} \in j\dot{X}$.” If this fails, then there is $\tilde{q} \leq [H]^j$ such that $\tilde{q} \Vdash_{jQ}$ “$\check{\lambda} \notin j\dot{X}$.” Express Q' as $Q * R$, and as above find a subiteration embedding of Q' into $(jP_\lambda)^\lambda$ that extends the embedding of Q, giving $H' \in \mathrm{Gen}_j(Q', G^*)$ such that $H' \restriction Q = H$, and $q' \in H'$. In other words if $\pi_1 : l(Q) \to \lambda^+$ embeds Q into $(jP_\lambda)^\lambda$ then we obtain $\pi_2 : l(Q') \to \lambda^+$ embedding Q' into $(jP_\lambda)^\lambda$ such that $\pi_1(\delta) = \pi_2(\alpha_\delta)$ for $\delta < l(Q)$. Now $j\pi$ embeds jQ into jQ' via $j\langle \alpha_\delta : \delta < l(Q) \rangle$, thus $(j\pi)(\tilde{q}) \in jQ'$ and

$$\mathrm{supp}\left((j\pi)(\tilde{q})\right) \subseteq j(\{\alpha_\delta : \delta < l(Q)\}).$$

Moreover $\operatorname{supp}([H']^j) = j''l(Q')$, if $\alpha < l(Q')$ then either $\alpha \in \{\alpha_\delta \colon \delta < l(Q)\}$, and then $(j\pi)(\tilde{q})(j\alpha)$ extends $[H']^j(j\alpha)$, or $\alpha \notin \{\alpha_\delta \colon \delta < l(Q)\}$, then $j(\alpha) \notin \operatorname{supp}((j\pi)(\tilde{q}))$. Consequently $(j\pi)(\tilde{q})$ and $[H']^j$ are compatible. But $[H']^j \Vdash_{jQ'}$ "$\check{\lambda} \in j(\pi\dot{X})$," while $(j\pi)(\tilde{q}) \Vdash_{jQ'}$ "$\check{\lambda} \notin j(\pi\dot{X})$" — a contradiction.

(ii) Each F^Q_j is obviously proper and contains $\operatorname{Club}(\kappa)$. Let $P_\kappa * Q \Vdash$ "$\dot{S} \subseteq \kappa$ is $F^Q_{\kappa,\gamma}$-positive" for some $\gamma < \Theta$ (or $\dot{S} \subseteq \operatorname{Sing}(\kappa)$ is stationary). We wish to prove that $P_\kappa * Q \Vdash$ "$\operatorname{Tr}(\dot{S}) \in F^Q_j$." Assume towards a contradiction that G^* is jP_κ-generic over V, $H \in \operatorname{Gen}_j(Q, G^*)$, and $[H]^j \nVdash_{jQ}$ "$\kappa \in j(\operatorname{Tr}(\dot{S}))$." So there is an H^* jQ-generic over $V[G^*]$ with $[H]^j \in H^*$ so that

$$V[G^* * H^*] \models \text{“}\dot{S} \text{ is nonstationary.”}$$

Since $(jP_\kappa)_{\kappa+1, j\kappa} * jQ$ is essentially κ-closed and Q_κ is κ^+-c.c. there is a sufficiently large $\alpha < \kappa^+$, such that if $G = G^* \restriction P_\kappa$, $\tilde{H} = G^* \restriction (Q_\kappa \restriction \alpha)$ then

$$V[G * \tilde{H}] \models \text{“}\dot{S} \text{ is nonstationary,”}$$

which is a contradiction with (i) as $V[G * H] \models$"$\dot{S}$ is $F^H_{\kappa,\gamma}$-positive" and Q is an subiteration of $Q_\kappa \restriction \alpha$ giving H from $\tilde{H}$ (provided α is large enough).

(iii) Assume that

$$P_\kappa * Q \Vdash \text{“}\dot{S} \subseteq \operatorname{Reg}(\kappa) \text{ and } \forall\gamma < \Theta : \dot{S} \text{ is } F^Q_{\kappa,\gamma}\text{-thin.”}$$

We want to prove that $P_\kappa * Q \Vdash$ "$\kappa \setminus \operatorname{Tr}(\dot{S}) \in F^Q_j$." Assume G^* is jP_κ-generic, $H \in \operatorname{Gen}_j(Q, G^*)$, $H^* \ni [H]^j$ jQ-generic over $V[G^*]$ and $V[G^* * H^*] \models$ "$\kappa \notin j(\kappa \setminus \operatorname{Tr}(\dot{S}))$," i.e. $V[G * \tilde{H}] \models$"$\dot{S}$ is stationary" where $\tilde{H} = (G^*) \restriction Q_\kappa$. But a club had been shot through $\kappa \setminus \dot{S}$ in the iteration Q_κ - a contradiction. ∎

LEMMA 2: *Let* Q_o *be an iteration of* $\langle CU(\dot{X}_\alpha) \colon \alpha < l(Q_o)\rangle$ *of order* Θ, $\dot{X}$ *a* $P_\kappa * Q_o$*-name for a subset of* κ, $p * q \in P_\kappa * Q_o$. *Then* Q_o *is an iteration of order* $\Theta + 1$ *w.r.t.* F_j, *and moreover if* $p * q \Vdash$"$\dot{X}$ *is* $F^{Q_o}_{\kappa,\gamma}$*-thin for all* $\gamma < \Theta$" *then* $p * q \Vdash$ "$\dot{X}$ *is* $F^{Q_o}_j$*-thin.*"

Proof: Assume towards a contradiction that $p * q \Vdash$ "$\dot{X}$ is $F^{Q_o}_j$-positive." Then we claim that the construction of filter systems $F_{\kappa,\gamma}$ in $M = \operatorname{Ult}(V, U)$ could not stop at Θ. F_j cannot be constructed in M, but we can construct its approximation.

Firstly define $\tilde{F}_{\kappa,\Theta}$ as follows:

Let $\tilde{F}^{\emptyset}_{\kappa,\Theta}$ $(Q = \emptyset)$ be generated in $V(P_\kappa)$ by all sets that should be there by (ii) and (iii), and by $\dot{X}_0$. Note that $\dot{X}_\alpha$ is forced to be in $F_j^{Q_o \restriction \alpha}$ for all $\alpha < l(Q)$ by the induction hypothesis. Hence $\tilde{F}^{\emptyset}_{\kappa,\Theta} \subseteq F^{\emptyset}_j$ verifying that $\tilde{F}^{\emptyset}_{\kappa,\Theta}$ is a proper filter. Similarly define $\tilde{F}^{Q}_{\kappa,\Theta}$ for iterations Q of order $\Theta + 1$ w.r.t. previously defined $\tilde{F}^{Q \restriction \alpha}_{\kappa,\Theta}$. We also have to make sure that $\dot{X}_\alpha \in \tilde{F}^{Q_o \restriction \alpha}_{\kappa,\Theta}$ for all $\alpha < l(Q_o)$. This filter system satisfies (ii) and (iii). Clearly $\tilde{F}^{Q'}_{\kappa,\Theta} \subseteq \tilde{F}^{Q}_{\kappa,\Theta}$ if Q' is an subiteration of Q, but (i) does not have to hold. To achieve that define

$$F^{\emptyset}_{\kappa,\Theta} = \bigcup \{\tilde{F}^{Q}_{\kappa,\Theta} \cap V(P_\kappa)\colon\ Q \text{ is an iteration of order } \Theta + 1 \text{ w.r.t. } \tilde{F}_{\kappa,\Theta}\ \}.$$

Then for Q an iteration of order $\Theta + 1$ w.r.t. previously defined $F^{Q \restriction \alpha}_{\kappa,\Theta}$'s by induction on $l(Q)$ define

$$F^{Q}_{\kappa,\Theta} = \bigcup \{\tilde{F}^{Q'}_{\kappa,\Theta} \cap V(P_\kappa * Q)\colon\ Q' \text{ is an iteration of order } \Theta + 1 \text{ w.r.t. } \tilde{F}_{\kappa,\Theta} \\ \text{such that } Q \text{ is an subiteration of } Q'\}.$$

It is not difficult to see that such Q' exists. We have constructed a filter system $F_{\kappa,\Theta}$ in M that satisfies (i)–(iii). Moreover Q_o is an iteration of order $\Theta + 1$ w.r.t. $F_{\kappa,\Theta}$, and so (iv) holds for the $\dot{X}$, $p * q$ from the assumption of the lemma — a contradiction. ∎

Let $G * H$ be $P_\kappa * Q_\kappa$-generic over V.

Lemma 3: $V[G * H] \models$ *"Full Reflection holds up to κ."*

Proof: For $\gamma < \Theta$ define $F^{H}_{\kappa,\gamma} = \bigcup_{\alpha<\kappa^+} F^{H \restriction \alpha}_{\kappa,\gamma}$. We know that $F^{H}_{\kappa,\gamma} \supseteq \mathrm{Club}(\kappa)$ is proper. By (i) if $S \in V[G * H \restriction \alpha]$ is $F^{H \restriction \alpha}_{\kappa,\gamma}$-positive then it is $F^{H}_{\kappa,\gamma}$-positive. Moreover by the construction $S \subseteq \mathrm{Reg}(\kappa)$ is stationary iff S is $F^{H}_{\kappa,\gamma}$-positive for some $\gamma < \Theta$ iff S is $F^{H \restriction \alpha}_{\kappa,\gamma}$-positive whenever $S \in V[G * H \restriction \alpha]$. Let us firstly prove that $V[G * H] \models$ "$S < \mathrm{Reg}(\kappa)$" for $S \subseteq \mathrm{Sing}(\kappa)$ stationary in $V[G * H]$. Let $S \in V[G * H \restriction \alpha]$ so that S is also stationary in this model, and so by (ii) $\mathrm{Tr}(S) \in F^{H \restriction \alpha}_{\kappa,\gamma}$ for all $\gamma < \Theta$, and consequently a club has been shot through $\mathrm{Sing}(\kappa) \cup \mathrm{Tr}(S)$.

Now let $S \subseteq \mathrm{Reg}(\kappa)$ be stationary, and γ_S the least γ such that S is $F^{H}_{\kappa,\gamma}$-positive. The following claim completes the proof of Full Reflection at κ in $V[G * H]$ (the proof for $\lambda < \kappa$ is identical).

CLAIM: *Let $S, T \subseteq \operatorname{Reg}(\kappa)$ be two stationary sets. Then $\gamma_S < \gamma_T$ iff $S < T$. Consequently $\gamma_S = \gamma_T$ iff $o(S) = o(T)$.*

Proof: Let $S, T \in V[G * H \restriction \alpha]$, $\gamma_S < \gamma_T$. Then S is $F^{H\restriction\alpha}_{\kappa,\gamma_S}$-positive, and so by (ii) $\operatorname{Tr}(S) \in F^{H\restriction\alpha}_{\kappa,\delta}$ for all $\delta > \gamma_S$. Thus $T \setminus \operatorname{Tr}(S)$ is $F^{H\restriction\alpha}_{\kappa,\delta}$-thin for all $\delta < \Theta$, and so a club has been shot through $\kappa \setminus (T \setminus \operatorname{Tr}(S))$, which means that $T \setminus \operatorname{Tr}(S)$ is nonstationary in $V[G * H]$, i.e. $S < T$.

On the other hand assume that $S < T$, so that necessarily $\gamma_S \leq \gamma_T$. By the definition of γ_S the set S is $F^{H\restriction\alpha}_{\kappa,\delta}$-thin for all $\delta < \gamma_S$, and so by (iii) $\operatorname{Tr}(S)$ is $F^{H\restriction\alpha}_{\kappa,\gamma_S}$-thin. Since $T \setminus \operatorname{Tr}(S)$ is nonstationary in $V[G * H]$, it must be $F^{H\restriction\alpha}_{\kappa,\gamma_S}$-thin. Thus $T = (T \setminus \operatorname{Tr}(S)) \cup \operatorname{Tr}(S)$ is $F^{H\restriction\alpha}_{\kappa,\gamma_S}$-thin, proving $\gamma_S < \gamma_T$.

Finally if $\gamma_S = \gamma_T$ and say $o(S) < o(T)$ there must be $S' < T$ such that $o(S) = o(S')$. By the fact proven above $\gamma_{S'} < \gamma_T = \gamma_S$, and so $S' < S$ — a contradiction. ∎

Finally we need to prove that $P_{\kappa+1}$ preserves large cardinal properties of κ. Let us firstly consider measurability and supercompactness of κ.

LEMMA 4: *Let $\lambda \geq \kappa$ be a cardinal such that*

(i) $V \cap {}^{\lambda}M \subseteq M$,

(ii) $\lambda^+ < j(\kappa) < j(\kappa^+) < \lambda^{++}$,

(iii) *there is no Mahlo cardinal between κ and $\lambda + 1$.*

Then the embedding $j\colon V \to M$ can be extended to $j^{**}\colon V[G * H] \to M[G^* * H^*]$ in $V[G * H]$ so that $V[G * H] \cap {}^{\lambda}M[G^* * H^*] \subseteq M[G^* * H^*]$.

Proof: By the definition of $P_{\kappa+1}$ the forcing $jP_{\kappa+1}$ factors as $P_{\kappa+1} * R_o * j(Q_\kappa)$. So all we need is to find an $R_o * j(Q_\kappa)$-generic filter $H_o * H^*$ over $M[G * H]$ so that $p * q \in G * H$ implies $j(p * q) \in G * H * H_o * H^*$. The factor iteration $R_o = (jP_{\kappa+1})_{\kappa+1, j\kappa}$ starts with a nontrivial forcing at the first Mahlo cardinal in M above κ which must be above λ. Consequently R_o is essentially λ-closed in $M[G * H]$ as well as in $V[G * H]$. Let D be a λ-closed dense subset of R_o. The number of dense subsets of D in $M[G * H]$ is $j(\kappa^+)$ and the cardinality of $j(\kappa^+)$ in V is just λ^+. Thus we have only λ^+ dense subsets of a forcing that is λ-closed in $V[G * H]$, and so it is easy to construct $H_o \in V[G * H]$ that is R_o-generic over $M[G * H]$. Obviously $p \in G$ implies $j(p) \in G^* = G * H * H_o$, so that j extends to $j^*\colon V[G] \to M[G^*]$ in $V[G * H]$. It immediately follows from the κ-c.c. of P_κ that $V[G] \cap {}^{\lambda}M[G^*] \subseteq M[G^*]$. Next we need to find a filter $H^* \in V[G * H]$ that is

$j^*(Q_\kappa)$-generic over $M[G^*]$, and such that $[H \restriction \alpha]^j \in H^*$ for all $\alpha < \kappa^+$. Notice that $[H \restriction \alpha]^j$ is a good condition in $j^*(Q_\kappa \restriction \alpha)$ by Lemma 2.

It is easy to see that the number of antichains of Q_κ (in $V[G]$) is only κ^+: if $A \subseteq Q_\kappa$ is an antichain, then $|A| \leq \kappa$, which implies that there is an $\alpha < \kappa^+$ such that $A \subseteq Q_\kappa \restriction \alpha$; but the number of subsets of $Q_\kappa \restriction \alpha$ is only κ^+. By elementarity $M[G^*] \models$"the number of antichains in $j^*(Q_\kappa)$ is $j(\kappa^+)$". Moreover $M[G^*] \models$"$j^*(Q_\kappa)$ is essentially λ-closed". Let D be a λ-closed dense open subset of $j^*(Q_\kappa)$, put

$$\mathcal{D} = \{A \in M[G^*]: A \subseteq D \text{ is an antichain}\}.$$

Then $V[G{*}H] \models$ "D is λ-closed, $|\mathcal{D}| = |j(\kappa^+)| = \lambda^+$." Now we have to distinguish two cases: if $\lambda \geq \kappa^+$ then $[H]^j = \cup_{\alpha<\kappa^+}[H \restriction \alpha]^j$ is a good master condition in $j^*(Q_\kappa)$, and we can easily build up $H^* \in V[G * H]$ $j^*(Q_\kappa)$-generic over $M[G^*]$ such that $[H]^j \in H^*$. If $\lambda = \kappa$ then we have to be more careful. Let $\langle A_\alpha: \alpha < \kappa^+\rangle$ be an enumeration of $\mathcal{D}$ in which each element of $\mathcal{D}$ occurs cofinally many times. Construct a descending sequence of conditions $\langle q_\alpha: \alpha < \kappa^+\rangle \subseteq D$ with the following properties

(i) $q_\alpha \in j^*(Q_\kappa \restriction \alpha)$,

(ii) $q_\alpha \leq [H \restriction \alpha]^j$,

(iii) if $A_\alpha \subseteq j^*(Q_\kappa \restriction \alpha)$ then q_α strengthens a condition in A_α.

The sequence $\langle q_\alpha: \alpha < \kappa^+\rangle$ generates a j^*Q_κ-generic filter $H^* \in V[G * H]$ over $M[G^*]$ such that each $[H \restriction \alpha]^j$ is in H^*.

Since $P_{\kappa+1}$ is κ^+-c.c. each $P_{\kappa+1}$-name for a λ-sequence of ordinals in V is already in M. Hence $V[G * H] \cap {}^\lambda M[G^* * H^*] \subseteq M[G^* * H^*]$. ∎

By the lemma if κ is measurable, or λ-supercompact with no Mahlo cardinal between κ and $\lambda+1$, and if $P_{\kappa+1}$ is constructed using a corresponding elementary embedding j, then the forcing preserves measurability, or λ-supercompactness of κ.

Now suppose κ is supercompact. We can assume without loss of generality that there is no inaccessible cardinal above κ, cutting off the universe if there is any. For each $\lambda > \kappa$ there is a λ-supercompact embedding j given by an ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Assign to λ a forcing $P^\lambda_{\kappa+1}$ constructed from j as above. It is easy to estimate the number of possible forcings $P_{\kappa+1}$ to be $\leq \kappa^{++}$. Consequently there is a proper class of λ's with the same $P_{\kappa+1} = P^\lambda_{\kappa+1}$. This $P_{\kappa+1}$ preserves the λ-supercompactness of κ for any of those λ's, and so the supercompactness of κ.

Let us turn our attention to strong cardinals. The following is essentially the idea for how to modify the construction above.

LEMMA 5: *Let $j\colon V \to M$ be given by a (κ, λ)-extender:* $\mathrm{crit}(j) = \kappa$, $V \cap {}^{\kappa}M \subseteq M$, $M = \{(jf)(a)\colon a \in [\lambda]^{<\omega}, f \in {}^{[\kappa]^{|a|}}V\}$. *Moreover assume that P is a notion of forcing such that $M \models$ "$|P| \leq j(\kappa^{+})$, P has $j(\kappa^{+})$-c.c., and P is λ-closed." Then there is $G \in V$ P-generic over M.*

Proof (J. Zapletal): We can assume that $P \subseteq j(\kappa^{+})$. Let $\langle f_{\alpha}\colon \alpha < \kappa^{+}\rangle$ be an enumeration of all functions $\kappa \to [\kappa^{+}]^{\kappa}$. Construct a sequence $\langle p_{\alpha}\colon \alpha < \kappa^{+}\rangle$ of conditions in P as follows: Put $p_0 = 1$. For limit α get a lower bound of $\langle p_{\delta}\colon \delta < \alpha\rangle$ using closedness of M and P. For $\alpha = \beta + 1$ put $X = \{(jf_{\beta})(a)\colon a \in [\lambda]^{<\omega}$, $(jf_{\beta})(a) \subseteq P$ is a maximal antichain$\}$. X is a set in M of cardinality $\leq \lambda$, hence we can find $p_{\beta+1} < p_{\beta}$ that meets all of those maximal antichains using closedness of P in M.

By the chain condition the filter G generated by $\langle p_{\alpha}\colon \alpha < \kappa^{+}\rangle$ is P-generic over M. ∎

Let $j\colon V \to M$ be γ-strong, i.e. $\mathrm{crit}(j) = \kappa$, $V_{\kappa+\gamma} \subseteq M$, $\gamma < j(\kappa)$. It is a standard fact on extenders (see [Ka93]) that we can assume

$$M = \{(jf)(a)\colon a \in [\lambda]^{<\omega}, f \in {}^{[\kappa]^{|a|}}V\},$$

where $\lambda = |V_{\kappa+\gamma}|^{+M} < j(\kappa)$.

Assume there is no Mahlo cardinal between κ and $\lambda + 1$. Let $P_{\kappa+1}$ be constructed from j, $jP_{\kappa+1} = P_{\kappa+1} * R_o * (jQ_{\kappa})$, and $G * H$ $P_{\kappa+1}$-generic over V. To construct $H_o \in V[G * H]$ R_o-generic over $M[G * H]$ consider an enumeration $\langle f_{\alpha}\colon \alpha < \kappa^{+}\rangle$ of all functions in V from κ to $[P_{\kappa}]^{\kappa}$. Construct a descending chain $\langle p_{\alpha}\colon \alpha < \kappa^{+}\rangle \subseteq R_o$ similarly as in the proof of lemma 5 so that p_{α} meets any maximal antichain $\subseteq R_o$ of the form $(jf_{\alpha})(a)/G*H$ ($a \in [\lambda]^{<\omega}$). We only have to observe that R_o is κ-closed in $V[G * H]$ and λ-closed in $M[G * H]$. The sequence $\langle p_{\alpha}\colon \alpha < \kappa^{+}\rangle$ generates a filter $H_o \subseteq R$ generic over $M[G * H]$. Now $j\colon V \to M$ is lifted to $j^{*}\colon V[G] \to M[G^{*}]$ in $V[G * H]$, where $G^{*} = G * H * H_o$. The embedding j^{*} is obviously again given by an (κ, λ)-extender.

To construct a $j^{*}Q_{\kappa}$-generic$/M[G^{*}]$ filter $H^{*} \in V[G * H]$ consider an enumeration $\langle f_{\alpha}\colon \alpha < \kappa^{+}\rangle$ of all functions from κ into $[Q_{\kappa}]^{\kappa}$, each with cofinally many repetitions. We need $[H \restriction \alpha]^{j} \in H^{*}$ for all $\alpha < \kappa^{+}$, so construct a descending sequence $\langle p_{\alpha}\colon \alpha < \kappa^{+}\rangle \subseteq j^{*}Q_{\kappa}$ so that

(i) $p_\alpha \in j^*(Q_\kappa \restriction \alpha)$,

(ii) $p_\alpha \leq [H \restriction \alpha]^j$,

(iii) p_α meets every maximal antichain $\subseteq j^*(Q_\kappa \restriction \alpha)$ of the form $(j^* f_\alpha)(a)$ for an $a \in [\lambda]^{<\omega}$.

Since any maximal antichain in j^*Q_κ is actually an antichain in $j^*(Q_\kappa \restriction \alpha)$ for some $\alpha < \kappa^+$, the sequence generates a desired $H^* \in V[G*H]$ j^*Q_κ-generic over $M[G^*]$. Therefore j^* lifts to $j^{**}\colon V[G*H] \to M[G^* * H^*]$. Obviously $V[G*H] \cap {}^\kappa M[G^* * H^*] \subseteq M[G^* * H^*]$ as $P_{\kappa+1}$ is κ^+-c.c.

Let $\mathcal{P}^\delta(x)$ denote the iterated power set of $x : \mathcal{P}^0(x) = x$, $\mathcal{P}^{\delta+1}(x) = \mathcal{P}(\mathcal{P}^\delta(x))$, and $\mathcal{P}^\delta(x) = \cup_{\alpha<\delta}\mathcal{P}^\alpha(x)$ for δ limit.

To prove that j^{**} is γ-strong it is enough to show that $\mathcal{P}^{\gamma_1, V[G*H]}(\kappa^+) \subseteq M[G^* * H^*]$, where $\gamma = 1+\gamma_1$. For each $\delta < \gamma_1$ fix a bijection $\pi_\delta\colon \mathcal{P}^\delta(\kappa^+) \times P_{\kappa+1} \to \mathcal{P}^\delta(\kappa^+)$ that is in M. (We actually need $\langle \pi_\delta\colon \delta < \gamma_1\rangle \in M$.) Then for each element x of $\mathcal{P}^{\gamma_1, V[G*H]}(\kappa^+)$ use π_δ's to find a code in $\mathcal{P}^{\gamma_1}(\kappa^+) \subseteq M$ for its $P_{\kappa+1}$-name $\dot{x}$. Consequently the name $\dot{x}$ itself can be decoded in M, and so $x = i_{G*H}(\dot{x})$ is in $M[G*H] \subseteq M[G^* * H^*]$.

We say that κ is strong if it is γ-strong for every γ. As in the case of a supercompact cardinal we can assume without loss of generality that there is no inaccessible cardinal above κ, and then use the same argument to find $P_{\kappa+1}$ that works for class many γ's preserving the strongness of κ. That concludes our proof of the main theorem.

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